and thus will be odd. This is just like the question on that test—the product/quotient of an even function and an odd function is an odd function!

Interchange Identities

Another thing you've probably noticed, just from knowing the graphs of sine and cosine, is that they're the same—they've just got a slight horizontal shift. Why is this? The quick explanation (going back to the unit circle definition) is that circles have this nice radial symmetry, so that the x and y coordinates are changing at the same rate–the only difference is that the x-coordinate (cosine) starts at 1, and the y-coordinate (sine) starts at 0. Put differently, the x-coordinate is about $\pi/2$ radians ahead of where the y-coordinate is. Or:

$$\cos(\theta) = \sin(\theta + \pi/2)$$

Stated in the opposite way:

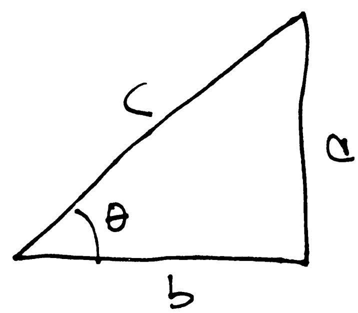
$$\sin(\theta) = \cos(\theta - \pi/2)$$

The Pythagorean Identity

There's another really cool relationship between trig functions. Namely: if I square the cosine of some angle, and then square the sine of the same angle, and add them, I just get one:

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

Why is this true? Imagine we have a right triangle, with base lengths a and b, hypotenuse c, and an angle θ:



Then we know that

$$\sin(\theta) = \frac{a}{c}$$

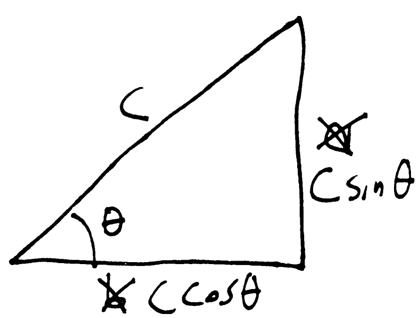
$$\cos(\theta) = \frac{b}{c}$$

Which, if we rearrange, is just another way of saying:

$$a = c\sin(\theta)$$

$$b = c\cos(\theta)$$

So we might as well label the sides of our triangle as $c\sin(\theta)$ and $c\cos(\theta)$, since they're just equal to the lengths:



But this is a right triangle, and so we must have:

$$\begin{aligned}
(c\sin\theta)^2 + (c\cos\theta)^2 &= c^2 \quad \text{(by the Pythagorean thm)}\\
c^2(\sin\theta)^2 + c^2(\cos\theta)^2 &= c^2 \quad \text{(distributing the square)}\\
(\sin\theta)^2 + (\cos\theta)^2 &= 1 \quad \text{(dividing by } c^2\text{)}
\end{aligned}$$

Often, by the way, we write things like $(\sin\theta)^2$ and $(\cos\theta)^2$ as $\sin^2\theta$ and $\cos^2\theta$, just as a more convenient notation (fewer parentheses!).

Note that we could write slightly modified versions of this identity. For example, if we divide both sides by $\cos^2(\theta)$, we get:

$$\begin{aligned}
\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta}\\
\tan^2\theta + 1 &= 1/\cos^2\theta
\end{aligned}$$

Or if we divide it all by $\sin^2\theta$:

$$\begin{aligned}
\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta}\\
1 + 1/\tan^2\theta &= 1/\sin^2\theta
\end{aligned}$$

For your convenience, I've summarized below all of the identities we've discussed. But please, please, don't try to memorize them! That will not help you understand trigonometry better! If you understand the trig—really *understand* it—then all of these identities[1] should make sense. They should be natural and obvious and you shouldn't even really have to think about these equations in order to apply them. This list, then, should not be a list of things you need to know, but a codification of things you already know.

Periodicity Identities (for any integer k):

- $\sin(\theta + 2k\pi) = \sin(\theta)$
- $\cos(\theta + 2k\pi) = \cos(\theta)$
- $\tan(\theta + k\pi) = \tan(\theta)$

Interchange Identities:

- $\cos(\theta) = \sin(\theta + \pi/2)$
- $\sin(\theta) = \cos(\theta - \pi/2)$

Symmetry Identities:

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$

Pythagorean Identity:

$\sin^2\theta + \cos^2\theta = 1$

[1]With the possible exception of the Pythagorean identity, which is not obvious and takes a little work to derive.