Extrema on an Interval

(extrema is the plural form of extremum ☺)

(extremum means point or value which is a max or min)

Definition:

Let f be defined on the interval I containing c.

1. f(c) is the minimum of f on I when $f(c) \leq f(x)$ for all x in I
2. f(c) is the maximum of f on I when $f(c) \geq f(x)$ for all x in I

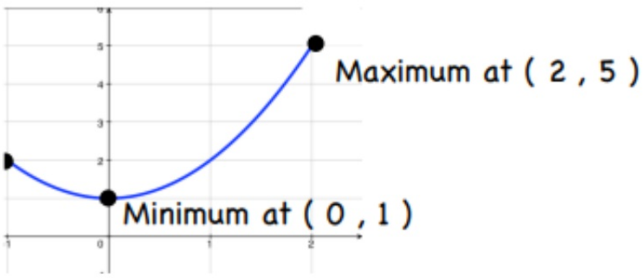
Extrema can occur at interior points or the endpoints of an interval

These are often called **absolute minimum/maximum**, or **global maximum/minimum**

Examples of Extrema

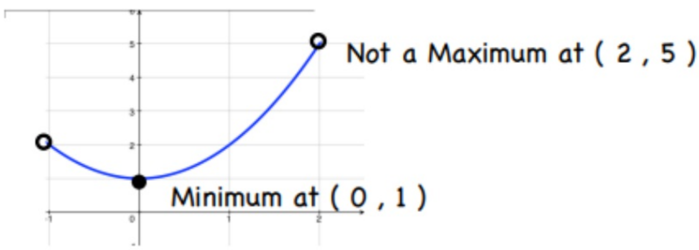
Example #1:

$f(x) = x^2 + 1$ on the closed interval [-1 , 2]



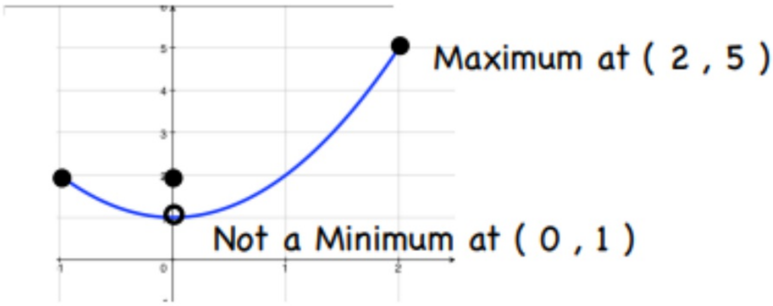
Example #2:

$f(x) = x^2 + 1$ on the open interval (-1 , 2)



Example #3:

$$g(x) = \begin{cases} x^2 + 1; & x \neq 0 \\ 2; & x = 0 \end{cases}$$ on the closed interval [-1 , 2]



The Extreme Value Theorem

If f is continuous on a closed interval [a, b], the f has both a minimum and a maximum on the interval.

The EVT is an existence theorem because it tells of the existence of a minimum and maximum values, but does not show how to find them.

Definition of Relative Extrema

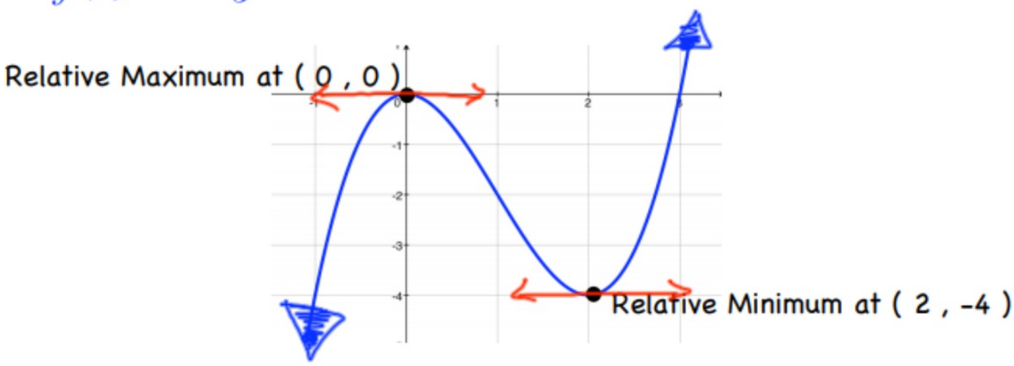
1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum of** f, or you can say that f has a relative maximum at the point (c , $f(c)$)

2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum of** f, or you can say that f has a relative minimum at the point (c , $f(c)$)

Examples of Relative Extrema

Example #1:

$f(x) = x^3 - 3x^2$

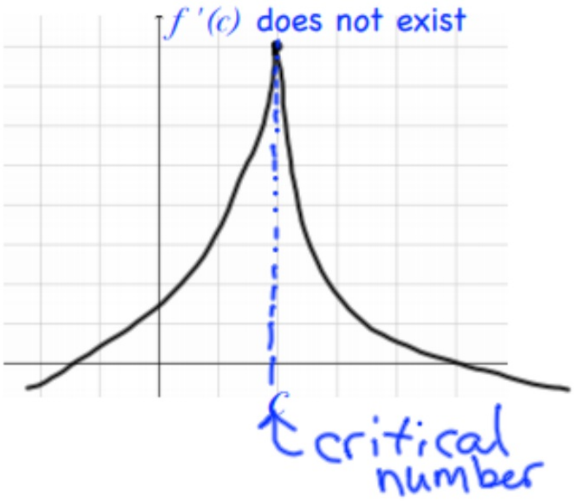


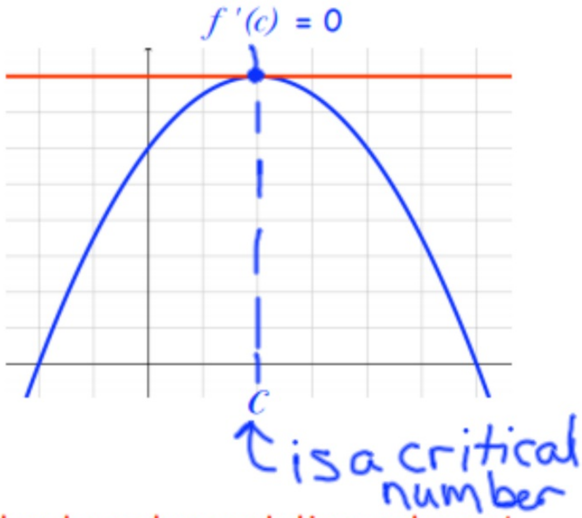
Think of **Relative Maximum** as a **"Hill"** and **Relative Minimum** as a **"Valley"**

If graph is smooth and rounded, the graph has a horizontal tangent line at the relative max or relative min If graph is sharp and peaked, the function is not differentiable at the relative max or relative min

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f





The critical numbers of a function need not produce relative extrema!

Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, the c is a critical number of f

Guidelines for Finding Extrema on a Closed Interval

1. Find the critical numbers of f in (a, b)

   Find the derivative and then find the "roots" of the derivative

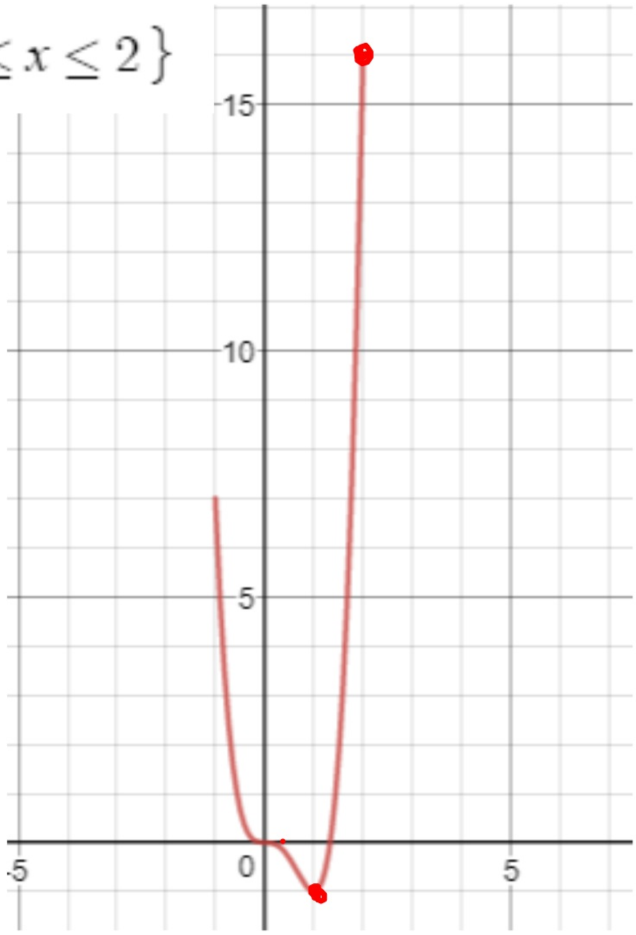
2. Evaluate f at each critical number in (a, b)

3. Evaluate f at each endpoint $[a, b]$

4. The least of these values is a mimimum.
The greatest is the maximum.

Example: Find the extrema of $f(x) = 3x^4 - 4x^3$ on the closed interval [-1 , 2].

$$y = 3x^4 - 4x^3 \ \{-1 \leq x \leq 2\}$$



Example: Find the extrema of $f(x) = \sin(x)$ on the closed interval [0 , 4π].

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

$$f(a) = f(b)$$

then there is at least one number c in (a, b) such that $f'(c)=0$.

In other words... **If f(a) = f(b)**

1) it guarantees the existence of an extreme value in the <u>interior</u> of a closed interval

2) there must be <u>at least one point</u> between a and b at which the derivative is 0

Rolle's Theorem applied if:

1. the function continuous on the closed interval
2. the function differentiable on the open interval
3. f(a) = f(b)

Apply Rolle's Thm:
1) Find *f'(x)*.
2) Set it equal to zero.
3) Solve for x.

Example:

a. Determine whether Rolle's Theorem can be applied to f on the closed interval [a,b].

b. If Rolle's Theorem can be applied, find all values of c in the open interval (a , b) such that f '(c) = 0.

$$f(x) = x^2 - 5x + 4 \qquad [1, 4]$$

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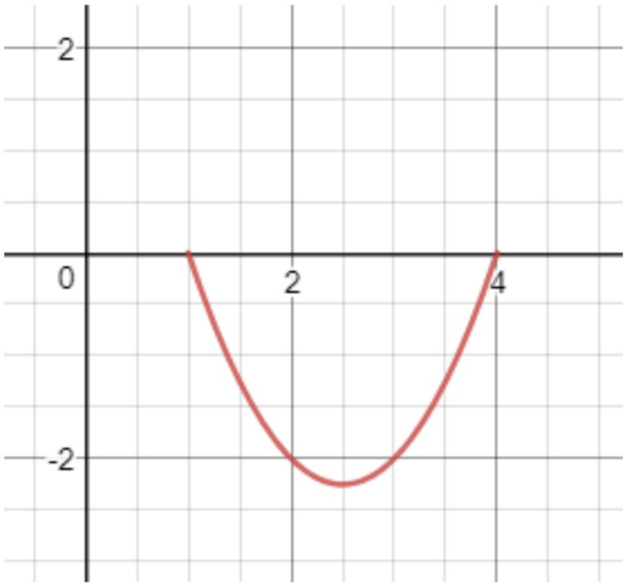


Mean Value Theorem

If f is continuous on the interval $[a, b]$ and differentiable on (a, b) then there exists a number c between a and b such that

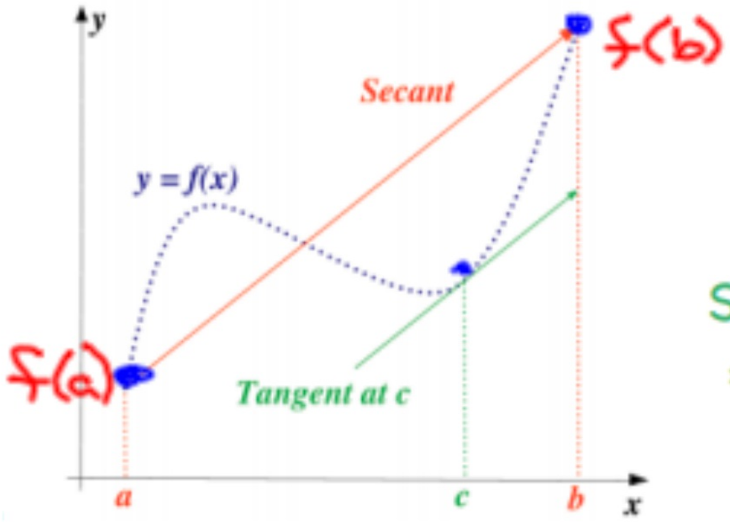
$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)$$

In other words...

1) it guarantees the existence of a tangent line that is parallel to the secant line through the points (a , $f(a)$) and (b , $f(b)$) → the endpoints of the interval

2) there must be a point on the open interval (a , b) at which the instantaneous rate of change is equal to the average rate of change over the interval [a , b]

$$\frac{y_1 - y_2}{x_1 - x_2}$$



Slope of t
= Slope t

Mean Value Theorem applied if:

1) the function continuous on the closed interval
2) the function differentiable on the open interval

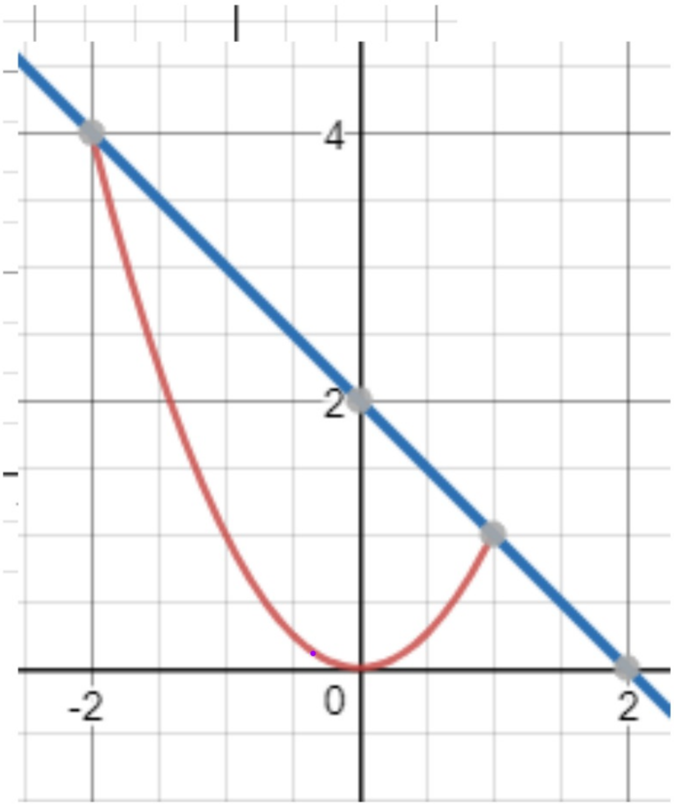
Apply MVT:

1. Plug a and b values into MVT formula to find slope of secant line.
2. Set f'(x) equal to slope of secant line.
3. Solve for x.

$$f(x) = x^2 \qquad [-2,1]$$

$$f'(c) = \frac{f(b}{b}$$

$$f(x) = x^2 \qquad [-2,1]$$



$f(x) = \sqrt{x-2}$ $[2,6]$

2 radar guns are set 5 miles apart on a highway. A truck passes the first patrol at 55 miles per hour. Four minutes later, the truck passes the 2nd patrol at 50 mph. Prove the truck must have exceeded the speed limit of 55 mph at some time during the 4 minutes.