SL 1
Dot Products WS #8

Name________________________

I THE SCALAR PRODUCT OF TWO VECTORS

SCALAR DOT PRODUCT

The **scalar product** of two vectors is also known as the **dot product** or **inner product**.

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, the **scalar product** of $\mathbf{v}$ and $\mathbf{w}$ is defined as

$\mathbf{v} \bullet \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$.

If $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

a $\mathbf{p} \bullet \mathbf{q}$

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a $\mathbf{p} \bullet \mathbf{q}$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= 2(-1) + 3(0) + (-1)2$$

$$= -2 + 0 - 2$$

$$= -4$$

EXERCISE 12I

1 For $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find:

a $\mathbf{q} \bullet \mathbf{p}$ **b** $\mathbf{q} \bullet \mathbf{r}$ **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$ **d** $3\mathbf{r} \bullet \mathbf{q}$

e $2\mathbf{p} \bullet 2\mathbf{p}$ **f** $\mathbf{i} \bullet \mathbf{p}$ **g** $\mathbf{q} \bullet \mathbf{j}$ **h** $\mathbf{i} \bullet \mathbf{i}$

2 For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** $\mathbf{b} \bullet \mathbf{a}$ **c** $|\mathbf{a}|^2$

d $\mathbf{a} \bullet \mathbf{a}$ **e** $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$ **f** $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

4 Find: **a** $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$ **b** $\mathbf{i} \bullet \mathbf{i}$ **c** $\mathbf{i} \bullet \mathbf{j}$

$$v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}|\cos\theta$$

$$\therefore \quad \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$

So, $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$ can be used to find the angle between two vectors **v** and **w**.

3 If $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** the angle between **p** and **q**.

If $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, find:

b the angle between **p** and **q**.

b
$$\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$$
$$\therefore \quad \cos\theta = \frac{\mathbf{p} \bullet \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$$
$$= \frac{-4}{\sqrt{4+9+1}\sqrt{1+0+4}}$$
$$= \frac{-4}{\sqrt{70}}$$
$$\therefore \quad \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right) \approx 119^o$$

5 Find $\mathbf{p} \bullet \mathbf{q}$ if: **a** $|\mathbf{p}| = 2,\ |\mathbf{q}| = 5,\ \theta = 60^o$ **b** $|\mathbf{p}| = 6,\ |\mathbf{q}| = 3,\ \theta = 120^o$

If **v** is perpendicular to **w** then $\theta = 90^o$.

$$\therefore \quad \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$
$$= |\mathbf{v}||\mathbf{w}|\cos 90^o$$
$$= 0$$

If **v** is parallel to **w** then $\theta = 0^o$ or 180^o.

$$\therefore \quad \mathbf{v} \bullet \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$$
$$= |\mathbf{v}||\mathbf{w}|\cos 0^o \quad \text{or} \quad |\mathbf{v}||\mathbf{w}|\cos 180^o$$
$$= \pm|\mathbf{v}||\mathbf{w}|$$

7 Find t given that these vectors are perpendicular:

a $\mathbf{p} = \begin{pmatrix} 3 \\ t \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

c $\mathbf{a} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2-3t \\ t \end{pmatrix}$ **d** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix}$

Find t such that $\mathbf{a} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ t \end{pmatrix}$ are perpendicular.

Since **a** and **b** are perpendicular, $\mathbf{a} \bullet \mathbf{b} = 0$

$$\therefore \quad \begin{pmatrix} -1 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ t \end{pmatrix} = 0$$
$$\therefore \quad (-1)(2) + 5t = 0$$
$$\therefore \quad -2 + 5t = 0$$
$$\therefore \quad 5t = 2 \quad \text{and so} \quad t = \tfrac{2}{5}$$

8 For question **7** find, where possible, the value(s) of t for which the given vectors are *parallel*. Explain why in some cases the vectors can never be parallel.

Level 8: Use y1 and y2 and find the intersection.

10 **a** Show that $\begin{pmatrix}1\\1\\5\end{pmatrix}$ and $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ are perpendicular.

b Find t if $\begin{pmatrix}3\\t\\-2\end{pmatrix}$ is perpendicular to $\begin{pmatrix}1-t\\-3\\4\end{pmatrix}$.

14 Find the measure of the angle between the lines:

a $x - y = 3$ and $3x + 2y = 11$ **b** $y = x + 2$ and $y = 1 - 3x$

c $y + x = 7$ and $x - 3y + 2 = 0$ **d** $y = 2 - x$ and $x - 2y = 7$

Find the measure of the angle between the lines $2x + y = 5$ and $3x - 2y = 8$.

$2x + y = 5$ has gradient $-\frac{2}{1}$ and $\therefore$ direction vector $\begin{pmatrix}1\\-2\end{pmatrix}$ which we call **a**.

(SLOPE)

$3x - 2y = 8$ has gradient $\frac{3}{2}$ and $\therefore$ direction vector $\begin{pmatrix}2\\3\end{pmatrix}$ which we call **b**.

If the angle between the lines is θ, then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{(1 \times 2) + (-2 \times 3)}{\sqrt{1+4}\sqrt{4+9}} = \frac{-4}{\sqrt{5}\sqrt{13}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-4}{\sqrt{65}}\right) \approx 119.7^o$$

When finding the angle between two lines we choose the acute angle, in this case $180^o - \theta$.

$\therefore$ the angle is about 60.3^o.

15 Find the form of all vectors which are perpendicular to:

a $\begin{pmatrix}5\\2\end{pmatrix}$ **b** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **c** $\begin{pmatrix}3\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-4\\3\end{pmatrix}$ **e** $\begin{pmatrix}2\\0\end{pmatrix}$

Find the form of all vectors which are perpendicular to $\begin{pmatrix}3\\4\end{pmatrix}$.

$\begin{pmatrix}3\\4\end{pmatrix} \bullet \begin{pmatrix}-4\\3\end{pmatrix} = -12 + 12 = 0$

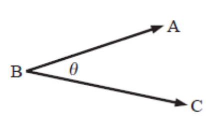
So, $\begin{pmatrix}-4\\3\end{pmatrix}$ is one such vector

The required vectors have the form $k\begin{pmatrix}-4\\3\end{pmatrix}$, $k \neq 0$.

16 Find the angle ABC of triangle ABC for A(3, 0, 1), B(−3, 1, 2) and C(−2, 1, −1).

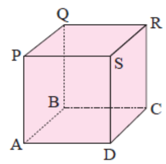
Hint: To find the angle at B, use $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

What angle is found if $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used?



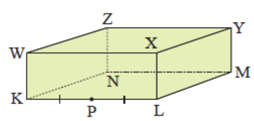
17 For the cube alongside with sides of length 2 cm, find using vector methods:

a the measure of angle ABS
b the measure of angle RBP
c the measure of angle PBS.

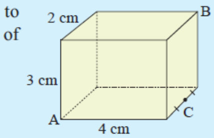


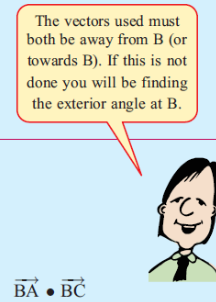
18 [KL], [LM] and [LX] are 8, 5 and 3 units long respectively. P is the midpoint of [KL]. Find, using vector methods:

a the measure of angle YNX
b the measure of angle YNP.



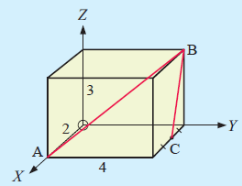
Use vector methods to determine the measure of angle ABC.





Placing the coordinate axes as illustrated,
A is (2, 0, 0), B is (0, 4, 3) and C is (1, 4, 0)

$\therefore$ $\overrightarrow{BA}$ is $\begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC}$ is $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$



$$\cos(\hat{ABC}) = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|}$$

$$= \frac{2(1)+(-4)(0)+(-3)(-3)}{\sqrt{4+16+9}\sqrt{1+0+9}}$$

$$= \frac{11}{\sqrt{290}}$$

$$\therefore\ \hat{ABC} = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.8^o$$

EXERCISE 12I

1 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

2 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

3 **a** -1 **b** 94.1^o **4** **a** 1 **b** 1 **c** 0

5 **a** 5 **b** -9

7 **a** $t = 6$ **b** $t = -8$ **c** $t = 0$ or 2 **d** $t = -\frac{3}{2}$

8 **a** $t = -\frac{3}{2}$ **b** $t = -\frac{6}{7}$ **c** $t = \frac{-1 \pm \sqrt{5}}{2}$ **d** impossible

9 Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} = 0$ **10** **b** $t = -\frac{5}{6}$

14 **a** 78.7^o **b** 63.4^o **c** 63.4^o **d** 71.6^o

15 **a** $k\begin{pmatrix}-2\\5\end{pmatrix}$, $k \neq 0$ **b** $k\begin{pmatrix}-2\\1\end{pmatrix}$, $k \neq 0$

c $k\begin{pmatrix}1\\3\end{pmatrix}$, $k \neq 0$ **d** $k\begin{pmatrix}3\\4\end{pmatrix}$, $k \neq 0$

e $k\begin{pmatrix}0\\1\end{pmatrix}$, $k \neq 0$

16 $\widehat{ABC} \approx 62.5^o$, the exterior angle 117.5^o

17 **a** 54.7^o **b** 60^o **c** 35.3^o

18 **a** 30.3^o **b** 54.2^o **19** **a** $M(\frac{3}{2}, \frac{5}{2}, \frac{3}{2})$ **b** 51.5^o

20 **a** $t = 0$ or -3 **b** $r = -2$, $s = 5$, $t = -4$